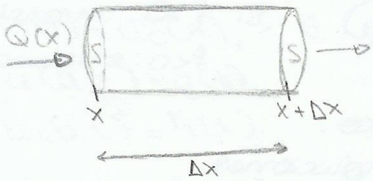
Modelización

13/Feb/2015

Tema 1.

1) Ecuación del calor:



Variación de calor: $\frac{\partial Q}{\partial t}$

$$\boxed{\frac{\partial Q}{\partial t} = \underbrace{Q(x)\cdot S - Q(x+\Delta x)\cdot S}_{\text{flujo de calor}} \quad (2)}$$

$$Q = \underbrace{\Delta m}_{\substack{\text{masa en}\\ [x,\,x+\Delta x]}} \cdot \overbrace{\underbrace{c}_{\text{calor específico}}}^{\text{cap. calorífica}} \cdot \underbrace{u}_{\text{temperatura}}$$

$C_{agua} = 1 \frac{kcal}{kg^{\circ}C}$ (el agua nos "roba" mucho más calor que el aire) $[C_{mármol} > C_{madera}]$

$C_{aire} = 0{,}24 \frac{kcal}{kg^{\circ}C}$

2) Derivo respecto de t (tiempo):

$$\frac{\partial Q}{\partial t} = \Delta m \cdot c \cdot \frac{\partial u}{\partial t} \quad (1)$$

De (1) y (2): $\Delta m \cdot c \cdot \frac{\partial u}{\partial t} = Q(x)\cdot S - Q(x+\Delta x)\cdot S \quad (3)$

Dividimos (3) entre $S\Delta x$ (volumen): $\underbrace{\rho}_{\text{densidad}} \cdot c \cdot \frac{\partial u}{\partial t} = \frac{Q(x) - Q(x+\Delta x)}{\Delta x} = -\frac{Q(x+\Delta x) - Q(x)}{\Delta x}$

3) Hacemos $\Delta x \to 0$: $\rho c \cdot \frac{\partial u}{\partial t} = -\lim_{\Delta x \to 0} \frac{Q(x+\Delta x) - Q(x)}{\Delta x} = -\frac{\partial Q}{\partial x}$; $\rho c \frac{\partial u}{\partial t} = -\frac{\partial Q}{\partial x}$

El calor va de ⊕ temp. a ⊖ temper.
El gradiente va de ⊖ a ⊕

Ley de Fourier: $Q = -\underbrace{k}_{\text{conductividad térmica}} \frac{\partial u}{\partial x}$ (1D) [En general $Q = -k \underbrace{\nabla u}_{\text{"grad" } u}$]

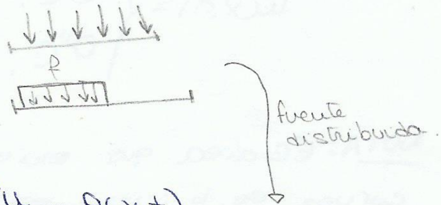
el calor fluye al contrario que el gradiente

$$\rho c \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) \Longrightarrow \boxed{\rho c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) = 0}$$

Ecuación del calor.

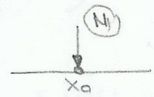
(*) Si k es cte en espacio: $\rho c \cdot \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = 0} \quad \text{donde} \quad \boxed{D = \frac{k}{\rho c}} \text{ difusividad térmica.}$$



Si tenemos términos fuente (o términos sumidero) $\longrightarrow \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = \underbrace{f(x,t)}_{\text{fuente de calor distribuida}}$.

Podemos tener una fuente puntual de calor M en cierto punto (x_0) $\longrightarrow \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = M \cdot \delta(x - x_0)$



En n-dimensiones: $\frac{\partial u}{\partial t} - D \underbrace{\Delta}_{\text{laplaciano}} u = M \delta(x - x_0) + f(x,t)$

En coordenadas cartesianas: $\frac{\partial u}{\partial t} - D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = M \delta(x - x_0) + f(x,t)$

Si D no es constante:

$$\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = M \delta(x - x_0) + f(x,t)$$

+ condiciones de contorno
+ condición inicial

20/Feb/2015

Remate tema 1:

Obtuvimos:

$$\rho c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) = f(x,t) + M\delta(x - x_p)$$

si $f>0$: introducimos calor.
$f<0$: extraemos calor

La ecuación del calor es de tipo parabólico.
La ecuación planteada tiene infinitas soluciones.

Ecuación del calor con una condición inicial:

$$\rho c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) = f(x,t) + M\delta(x - x_p) \Rightarrow$$

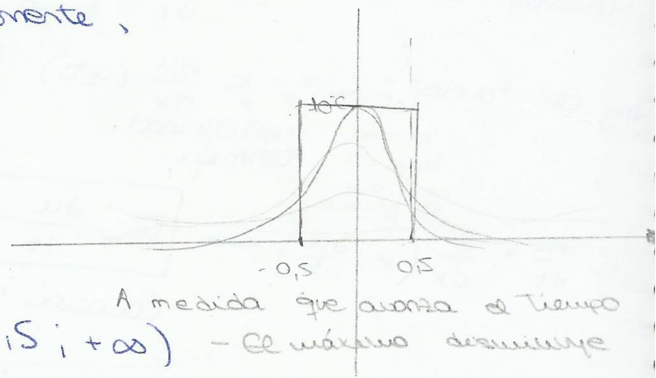
$$\Rightarrow \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(D \frac{\partial u}{\partial x}\right) = f + M\delta(x-x_p) \quad \left(\boxed{D = \frac{k}{\rho c}}\right)$$

$x \in (a,b)$ $t>0$.

$u(x,0) = u_0(x)$ $x \in (a,b)$
(cond. inicial)

Si suponemos que $a \to -\infty$, $b \to \infty$

La condición inicial interviene sólo al principio mientras que f se aplica constantemente.



Ejemplo:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\overset{D}{2} \frac{\partial u}{\partial x}\right) = 0 \quad x \in (-\infty, \infty)$$

$$u(x,0) = \begin{cases} 10°C, & x \in [0{,}5; 0{,}5] \\ 0°C, & x \in (-\infty; -0{,}5) \cup (-0{,}5; +\infty) \end{cases}$$

A medida que avanza el tiempo
- El máximo disminuye
- La curva se ensancha.
- La simetría se conserva.

NOTA: El área que encierra cada una de las curvas es la misma $\Rightarrow$ la cantidad total de calor $\left(Q = \rho c \int_{-\infty}^{\infty} u\,dx\right)$ se conserva en ausencia de fuentes de calor.

Habitualmente lo que pasa en los extremos es relevante, necesitamos introducir condiciones de contorno.

Tipos de condiciones de contorno

1.) Tipo DIRICHLET (Conocemos valor de la temperatura en extremos.)

2.) Tipo NEUMANN. (Conocemos el flujo de calor en extremos)

Problema de Dirichlet asociado a conducción de calor.

Problema de valores

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(D\dfrac{\partial u}{\partial x}\right) = f(x,t) \quad x \in (a,b),\ t>0 \\ \text{C. inicial: } u(x,0) = u_0(x),\ x \in (a,b) \\ \text{C. contorno: } \begin{cases} u(a,t) = g(t) \\ u(b,t) = h(t) \end{cases} \end{cases}$$

edp (pde)

Problema de Neumann asociado a conducción de calor

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) = f(x,t) \qquad x \in (a,b)\ t>0.$$

$$\text{C.I.: } u(x,0) = u_0(x) \qquad x \in (a,b). \qquad \boxed{D = \frac{k}{\rho c}}$$

$$\text{C.C.: } \begin{cases} -k\dfrac{\partial u}{\partial x}(a,t) = \varphi(t) \\ -k\dfrac{\partial u}{\partial x}(b,t) = \psi(t) \end{cases}$$

6/03/2015

CONTINUACIÓN TEMA 2:

Aplicaciones a mecánica de fluidos:

1. Conservación de masa.
2. Conservación de cantidad de movimiento
3. Conservación de energía.

1 Conservación de masa:

Flujo convectivo de materia → A, $V_A \rho_A$

Flujo convectivo de materia → B, $V_B \rho_B$ (densidad)

A ———— B ————→ x

masa en AB

$$\frac{\partial m}{\partial t} = V_A\rho_A - V_B\rho_B$$

$$m = \int_A^B \rho\, dx$$

$$\frac{\partial}{\partial t}\int_A^B \rho\, dx = V_A\rho_A - V_B\rho_B$$

$$\int_A^B \frac{\partial \rho}{\partial t}\, dx = -\int_A^B \frac{\partial \rho V}{\partial x}\, dx$$

$$\boxed{-\rho_B V_B + \rho_A V_A}$$

Igualamos integrandos

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho V)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0$$

(ley de conservación de materia (ecuación de continuidad))

NOTAS:

1. Si fluido incompresible $\rho = cte$

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial}{\partial x}(\rho V) = \rho\frac{\partial V}{\partial x} \Rightarrow \boxed{\frac{\partial V}{\partial x} = 0}$$

Ec continuidad fluidos incompresibles.

*NOTAS:

② Generalización a varias dimensiones:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \text{; En coord. cartesianas: } \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \; \frac{\partial}{\partial z}\right)\begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix} \qquad \vec{v}(u, v, w)$$

Si fluido incompresible $\nabla \cdot \vec{v} = 0$

② Conservación de cantidad de movimiento (o de momento)

$$\frac{\partial(\rho \vec{v})}{\partial t} + \frac{\partial}{\partial x}\left((\rho v) v\right) + \frac{\partial P}{\partial x} = 0 \Rightarrow \frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho v^2) + \frac{\partial P}{\partial x} = 0$$

P–presión.

③ Conservación de la energía.

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}((E + P) \cdot v) = 0 \quad \text{donde } E = \rho e + \frac{1}{2}\rho v^2 \text{ (gases)}$$

ρ: densidad

e: energía interna específica.

$e = \frac{P}{\rho(\gamma - 1)}$: ecuación de estado para gases ideales.

$\gamma = \frac{C_p}{C_v} = \frac{\text{calor específico pres cte}}{\text{calor específico vol cte}}$.

en aire $\gamma \approx 1{,}4$

coeficiente adiabático.

ECUACIONES DE EULER 1D.)

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \\ \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 + P) = 0 \\ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x}((E + P) v) = 0 \end{cases}$$

ρv ⇐

E

obtengo:

ρ → densidad

V → velocidad

P → presión

El sistema de Euler es hiperbólico y numéricamente se resuelve por volúmenes finitos

Ecs Euler se pueden generalizar a Navier-Stokes:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v + P) = \rho g + D \cdot \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x}((E + P) v) = \rho g V \end{cases}$$

Ecs NavierStokes

17/Feb/2015
(F. Michavila)

MODELIZACIÓN.

4. Diferencias finitas y elementos finitos:

⊛ 4.1. Planteamiento del problema modelo:

1D. $(0,1)$ $\quad (1)\begin{cases} -u''(x) + C(x)\cdot u(x) = f(x) & x \in (0,1) \\ \underline{u(0) = u(1) = 0} \end{cases}$

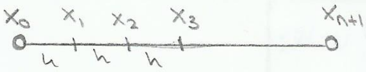
↳ condiciones de Dirichlet (homogéneas si $=0$)

viga: $C(x) = \dfrac{P}{EI}$

DATO:
$u(x_j) = u_j$
$C(x_j) = C_j$
$f(x_j) = f_j$

- Aproximar (1) por diferencias finitas:

$u(x) \rightarrow$ aproximado $u_h(x)$



$h = \dfrac{1}{n+1}$

$$u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u'''(x)\frac{h^3}{6} + O(h^4)$$
$$u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u'''(x)\frac{h^3}{6} + O(h^4)$$
error

$u(x+h) + u(x-h) = u(x) \simeq$
$\simeq \dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}$ (elimino el error)

(1) en $x = x_j$ $\qquad u(x_j - h) = u(x_{j-1}) = u_{j-1}$

$$-\frac{+u_{j+1} - 2u_j + u_{j-1}}{h^2} + C_j u_j = f_j \qquad j = 1, \dots, n$$

incógnitas: $u_1, u_2, \dots, u_n$

$u_0 = u_{n+1} = 0$.

$-\boxed{u_{j+1}} + (2 + C_j h^2)\boxed{u_j} - \boxed{u_{j-1}} = f_j h^2$

$$\begin{array}{cccc} j=1 & j=2 & \cdots & j=n \end{array}$$
$$\begin{pmatrix} 2+C_1h^2 & -1 & 0 & \cdots & 0 \\ -1 & 2+C_2h^2 & -1 & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & -1 & 2+C_nh^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

⊛ Para diferencias finitas, los trozos tienen que ser equidistantes.

⊛ Si no varía en el tiempo → estacionario.

⊛ Hacer ejercicio propuesto 1.

⊛ Método de Galerkin: (4.2)

1ª ETAPA

$$(1)\begin{cases} -u''(x) + C(x)u(x) = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$v \in C^1[0,1]$ $\quad v(0) = v(1) = 0$ función test.

continua ella y su 1ª derivada.

- Multiplicamos y lo integramos:

$$-\underbrace{\int_0^1 u''(x)\,v(x)\,dx}_{⊛\ \text{partes}} + \int_0^1 C(x)u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

⊛ partes:
$u''(x)dx = dv \longrightarrow v = u'(x)$
$v(x) = u \longrightarrow du = v'dx$
$\int u\,dv = v\cdot u - \int v\,du$
$[v(x)\cdot u'(x)]_0^1 - \int_0^1 u'(x)v'(x)\,dx$

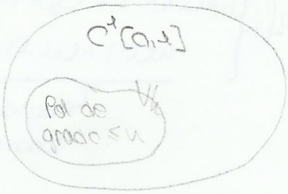
$$\int_0^1 u'(x)v'(x)\,dx - u'(1)\cancel{v(1)}^{0} + u'(0)\cancel{v(0)}^{0} + \int_0^1 C(x)u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$$(2)\quad \int_0^1 u'(x)v'(x)\,dx + \int_0^1 C(x)u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$ [Ya no tenemos derivadas segundas].

(2) "Hallar $u \in C^1[0,1]$, $u(0)=u(1)=0$ tal que cumple (2). $\forall v \in C^1[0,1]$; $v(0)=v(1)=0$"

(1) problema fuerte $\longrightarrow$ (2) problema débil

2ª ETAPA



(*) Hacer ejercicio propuesto 2.

- Aproximación de Galerkin:

$V_h \subset C^1[0,1]$ $\quad v(0)=v(1)=0$

(Pol : $\{1, x, \dots, x^{n-1}\}$)

$\dim(V_h)=n$ $\quad$ Base $\{\varphi_1, \dots, \varphi_n\}$

$$\boxed{u_h = \sum_{i=1}^{n} u_i \varphi_i}$$

- "Hallar $u_h \in V_h$; $u_h(0)=0$; $u_h(1)=0$ $\quad (2)_h \quad \int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)$

$\forall v_h \in V_h$ $\quad v_h(0)=v_h(1)=0$"

$v_h = \varphi_j \quad j=1,\dots,n$ $\Big\}$ tantas ecuaciones como la dimensión del espacio sea.

$$\sum_{i=1}^{n}\left[\int_0^1 (\varphi_i'\varphi_j' + c(x)\varphi_i\varphi_j)dx\right]u_i = \int_0^1 f\varphi_i dx \quad j=1,\dots,n \quad (3)$$

$A_{ij} = \int_0^1 (\varphi_i'\varphi_j' + c(x)\varphi_i\varphi_j)dx \longrightarrow N^{os} \longrightarrow$ coeficientes

$F_j = \int_0^1 f\varphi_j dx \longrightarrow N^{os} \longrightarrow$ términos independientes.

u_i no conocidos

MÉTODO DE GALERKIN

$$(4) \quad \boxed{\sum_{i=1}^{n} A_{ij}u_i = F_{ij} \quad j=1,\dots,n}$$

sistema de n ecuaciones lineales

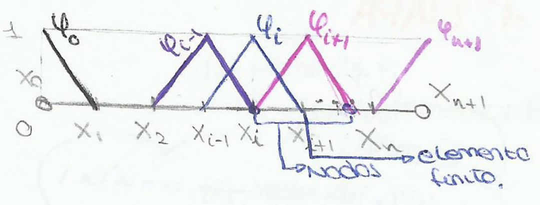
simétrica

$$[A_{ij}]\{u\} = \{f\}.$$

(*) Hacer ejercicio propuesto 3.

(*) Técnica de métodos finitos: (4.3)

3ª ETAPA



$x_{i+1} - x_i = h \quad \forall i$

$$\varphi_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{en } [x_{i-1}, x_i] \\[2ex] \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{en } [x_i, x_{i+1}] \\[2ex] 0 & \text{resto} \end{cases}$$

$\varphi_i(x_i) = 1$

$\varphi_i(x_j) = 0 \quad i \neq j$

$$\varphi_0 = \begin{cases} \dfrac{x - x_1}{x_0 - x_1} & \text{en } [x_0, x_1] \\[2ex] 0 & \text{resto} \end{cases}$$

$$\varphi_{n+1} = \begin{cases} \dfrac{x - x_n}{x_{n+1} - x_n} & \text{en } [x_n, x_{n+1}] \\[2ex] 0 & \text{resto} \end{cases}$$

$$A_{ij} = \underbrace{\int_0^1 \varphi_i' \varphi_j' \, dx}_{①} + \underbrace{\int_0^1 c(x) \varphi_i \varphi_j \, dx}_{②}$$

① $\frac{1}{h^2} \cdot h + \left(-\frac{1}{h}\right)^2 \cdot h = \frac{2}{h}$,, $i = j$

$\left(-\frac{1}{h}\right) \cdot \frac{1}{h} \cdot h = -\frac{1}{h}$,, $|i-j| = 1$

0 $\quad |i-j| > 1$

① $\begin{cases} \frac{2}{h} & i = j \\ -\frac{1}{h} & |i-j| = 1 \\ 0 & \text{resto} \end{cases}$

② $\frac{h}{\cancel{2}} \cancel{2} c(x_j)$ $\quad i = j$

0 $\quad i \neq j$

$$\frac{1}{h} \cdot \begin{bmatrix} 2+h^2c_1 & -1 & \cdots & \\ -1 & 2+h^2c_2 & -1 & \cdots \\ & & \ddots & \\ \cdots & & -1 & 2+h^2c_n \end{bmatrix}$$

diferencias finitas es un caso particular de elementos finitos.

$\int_0^1 f(x) \varphi_j \, dx \quad f_j h \longrightarrow$ TC

(*) Ejercicio 3.bis. {Trapecio y rectángulo compuestos}

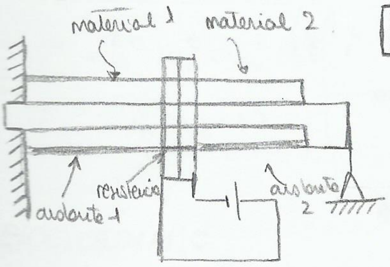
TC $\int_0^{x_n} f(x)\,dx \simeq \frac{h}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_n) + f(x_{n+1})]$

(*) Ejercicio 4.bis

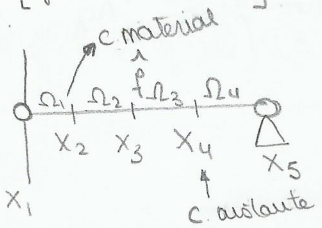
Buscar en Internet las biografías de Galerkin y Dirichlet y escribir una nota breve de una contribución matemática de cada uno de ellos.

4.4. Aplicación a un modelo unidimensional real... pero sencillo: 24/Feb/2015. (Muchavila)

$u = u(x)$ $\quad \sigma(x) = -k(x)\dfrac{du}{dx}$



$[\![f(x)]\!] = \lim_{\varepsilon \to 0} \left[f(x+\varepsilon) - f(x-\varepsilon)\right]$ → salto



• Formulación del problema: Dados $k_1, k_2, b(x), u, u_2$

$$-\frac{d}{dx}\left[k(x)\frac{du(x)}{dx}\right] + b(x)u = f(x) \quad \text{en } \Omega_i \quad i = 1,\dots,4$$

$$[\![k(x_i)\frac{du}{dx}(x_i)]\!] = 0 \qquad x_i = x_2, x_4$$

$$-[\![k(x_3)\frac{du}{dx}(x_3)]\!] = \hat{f} \qquad \begin{array}{l} u(0) = u_1 \\ \sigma(\ell) = p_\ell[u(\ell) - u_\ell] = -k(\ell)\dfrac{du}{dx}(\ell) \;\circledast \end{array}$$

$\circledast$ $-k(\ell)\dfrac{du}{dx}(\ell) - p_\ell[u(\ell)] = p_\ell u_\ell$ → Resolvemos → función test $v \in C^1[0,\ell]$ $v(0) = 0$

$$\int_0^\ell [-(ku')' + bu]v\,dx = \int_0^\ell f v\,dx$$

$$\underbrace{-\int_0^\ell (ku')'v\,dx}_{\circledast 1} + \int_0^\ell bu\,v\,dx = \int_0^\ell f v\,dx$$

$$\circledast 1 \; -\sum_{i=1}^{4}\int_{x_i}^{x_{i+1}} (ku')'v\,dx = \sum_{i=1}^{4}\left[-ku'v\Big|_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} ku'v'\,dx\right] =$$

$$= \int_0^\ell ku'v'\,dx + k\cancel{u'(x_1)}\,v(x_1) + \cancel{[\![ku'(x_2)]\!]}\,v(x_2) + [\![ku'(x_3)]\!]v(x_3) + \cancel{[\![ku'(x_4)]\!]}\,v(x_4) \underbrace{-ku'(\ell)}_{\sigma(\ell)}v(\ell) \overset{*}{=}$$

$x_1 = 0$

$$\overset{*}{=} \int_0^\ell ku'v'\,dx - \hat{f}v(x_3) + p_\ell[u(\ell) - u_\ell]v(\ell)$$

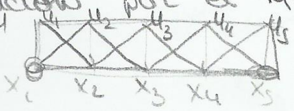
...ar $u \in H^1(0,\ell)$ $u(0) = u_1$ ...) ...ébil

$$\int_0^\ell (ku'v' + bu\,v)dx + p_\ell u(\ell)v(\ell) = \int_0^\ell f v\,dx + \hat{f}v(x_3) + p_\ell u_\ell v(\ell) \qquad \forall v \in H^1(0,\ell) \;\; v(0) = 0$$

cuadrado integrable

Ejercicio propuesto 4ter. Comprobar que se verifica que: $\displaystyle\int_a^b \delta(x-c)v(x)dx = v(c)$, $a < c < b$ / $\displaystyle\int_a^b \delta(x)v(x)dx = v(0)$, $a < 0 < b$

4.5. Aproximación por el MEF de 4.4



...2) ...debe H^h

$\{x_i\}_{i=1}^5$

$\dim(H^h) = 5$

$\{\varphi_i\}_{i=1}^5$

APROXIMADO

"Hallar $u_h \in H^h$, $u_h(0) = u_1$

$$\int_0^\ell (k\,u_h'v_h' + b\,u_h v_h)dx + p_\ell u_h(\ell)v_h(\ell) = \int_0^\ell f v_h\,dx + \hat{f}v_h(x_3) + p_\ell u_\ell v_h(\ell)$$

$\forall v_h \in H^h$ $v_h(0) = 0$"

$$u_h = \sum_{i=1}^{5} u_i \varphi_i \qquad v_h = \varphi_j \qquad j = 1,\dots,5$$

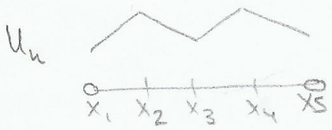
no es incógnita

$$\sum_{i=1}^{5} \underbrace{\left[\int_0^\ell (k\varphi_i'\varphi_j' + b\varphi_i\varphi_j)\,dx + P_\ell\, \varphi_i(\ell)\varphi_j(\ell) \right]}_{A_{ij}} u_i = \int_0^\ell f\varphi_j\,dx + \hat{f}\cdot\varphi_j(x_3) + P_\ell\, u_\ell\, \varphi_j(\ell) \qquad j = 1,\dots,5.$$

Sistema algebraico de ecuaciones lineales 5x5

$$[A_{ij}]\{u_i\} = \{f_j\}$$

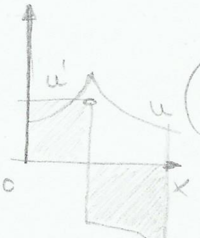
simétrica



3/MARZO/2015

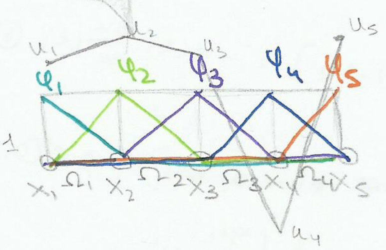
(*) Comentario:

$C^1[0,\ell]$



$H^1(0,\ell)$ $\begin{bmatrix} u \\ u' \end{bmatrix}$ cuadrado integrable $(u=0)$

Seguimos el problema:



$$H^h \qquad \{\varphi_i\}_{i=1}^{5}$$

$$\dim(H^h) = 5$$

$$\varphi_1 = \begin{cases} \dfrac{x - x_2}{x_1 - x_2} & [x_1, x_2] \\ 0 & \text{resto.} \end{cases} \qquad \varphi_2 = \begin{cases} \dfrac{x - x_1}{x_2 - x_1} & \text{en } (x_1, x_2] \\ \dfrac{x - x_3}{x_2 - x_3} & \text{en } [x_2, x_3] \\ 0 & \text{resto} \end{cases} \qquad \varphi_3, \varphi_4, \varphi_5 \dots$$

$$\varphi_h = \sum_{i=1}^{5} u_i\varphi_i \qquad v_h = \varphi_j \qquad j = 1,\dots,5$$

MATRIZ K (de rigidez) → simétrica salvo si hay derivada 2ª y 1ª (depende del problema físico)

$$\begin{bmatrix} \ddots & & 0 \\ & A_{ij} & \\ 0 & & \ddots \end{bmatrix} \qquad A_{ij} = K_{ij} = \int_0^\ell (k\varphi_i'\varphi_j' + b\varphi_i\varphi_j)\,dx + P_\ell\, u(\ell)\, v(\ell)$$

llena en banda

$$F_j = \int_0^\ell f\varphi_j\,dx + \hat{f}\varphi_j(x_3) + P_\ell\, u_\ell\, \varphi_j(\ell) \qquad \begin{bmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_5 \end{pmatrix} = \begin{pmatrix} F_j \end{pmatrix}$$

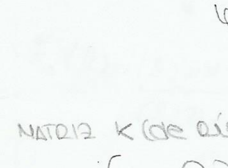
(*) Ejercicios propuestos 5,6 y 7

4.6.- Otra forma de hacer los cálculos:

local

FORMA shape f

$$\begin{cases} \varphi_1^e = 1 - \dfrac{x - x_1^e}{x_2^e - x_1^e} \\ \varphi_2^e = \dfrac{x - x_1^e}{x_2^e - x_1^e} \end{cases} \qquad u_h^e = u_1^e\varphi_1^e + u_2^e\varphi_2^e \qquad v_h^e = \varphi_j^e \quad j = 1,2$$



(*) Débil Localmente

$$\int_{x_1^e}^{x_2^e} (k u'v' + buv)\,dx = \int_{x_1^e}^{x_2^e} f v\,dx + \sigma(x_1^e)v(x_1^e) - \sigma(x_2^e)v(x_2^e)$$

Aproximación:

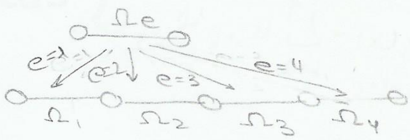
$$\int_{x_1^e}^{x_2^e} (k u_h' v_h' + b u_h v_h)\,dx = \int_{x_1^e}^{x_2^e} f v_h\,dx + \sigma(x_1^e)\, v_h(x_2^e) - \sigma(x_2^e)\, v_h(x_2^e)$$

$$\boxed{\begin{array}{l} u_h^e = u_1^e \psi_1^e + u_2^e \psi_2^e \\ v_h^e = \psi_j^e \quad j = 1,2 \end{array}} \longrightarrow \sum_{i=1}^{2} \left[\int_{x_1^e}^{x_2^e} (k \psi_i^{e\prime} \psi_j^{e\prime} + b \psi_i^e \psi_j^e)\,dx\right] u_i^e = \int_{x_1^e}^{x_2^e} f \psi_j^e\,dx + \sigma(x_1^e)\psi_j^e(x_1^e) - \sigma(x_2^e)\psi_j^e(x_2^e)$$

$j = 1,2.$

$$k_{ij}^e = \int_{x_1^e}^{x_2^e} (k \psi_i^{e\prime} \psi_j^{e\prime} + b \psi_i^e \psi_j^e)\,dx \qquad f_j^e = \int_{x_1^e}^{x_2^e} f \psi_j^e\,dx$$

$e = 1,2,3,4$
$$\begin{cases} k_{11}^e u_1^e + k_{12}^e u_2^e = f_1^e + \sigma(x_1^e) \\ k_{21}^e u_1^e + k_{22}^e u_2^e = f_2^e - \sigma(x_2^e) \end{cases}$$



$$e=1 \begin{cases} k_{11}^1 u_1 + k_{12}^1 u_2 = f_1^1 + \sigma(0) \\ k_{21}^1 u_1 + k_{22}^1 u_2 = f_2^1 - \sigma(x_2^-) \end{cases} \qquad e=3 \begin{cases} k_{11}^3 u_3 + k_{12}^3 u_4 = f_1^3 + \sigma(x_3^+) \\ k_{21}^3 u_3 + k_{22}^3 u_4 = f_2^3 - \sigma(x_4^-) \end{cases}$$

$$e=2 \begin{cases} k_{11}^2 u_2 + k_{12}^2 u_3 = f_1^2 + \sigma(x_2^+) \\ k_{21}^2 u_2 + k_{22}^2 u_3 = f_2^2 - \sigma(x_3^-) \end{cases} \qquad e=4 \begin{cases} k_{11}^4 u_4 + k_{12}^4 u_5 = f_1^4 + \sigma(x_4^+) \\ k_{21}^4 u_4 + k_{22}^4 u_5 = f_2^4 - \sigma(l) \end{cases}$$

elem. 2 elem 3

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & 0 \\ 0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & 0 \\ 0 & 0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & k_{12}^4 \\ 0 & 0 & 0 & k_{21}^4 & k_{22}^4 + P_l \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} f_1^1 + \sigma(0) \\ f_2^1 + f_1^2 + [\![\sigma(x_2)]\!] \\ f_2^2 + f_1^3 + [\![\sigma(x_3)]\!] \\ f_2^3 + f_1^4 + [\![\sigma(x_4)]\!] \\ f_2^4 + P_l u_l \end{pmatrix}$$

d2, d3 (*) (**)

$$[k] = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & 0 \\ 0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & 0 \\ 0 & 0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & k_{12}^4 \\ 0 & 0 & 0 & k_{21}^4 & k_{22}^4 \end{bmatrix}$$

C.I.

$[\![\sigma(x_2)]\!] = 0$

$[\![\sigma(x_4)]\!] = 0$

$[\![\sigma(x_3)]\!] = \hat{f}$

$$f \to \begin{pmatrix} f_1^1 + \sigma(0) \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 + \hat{f} \\ f_2^3 + f_1^4 \\ f_2^4 - \sigma(l) \end{pmatrix}$$

flujo $\to \sigma(l) = P_l u_5 - P_l u_l \to$ lo metemos en lugar de $\sigma(l)$ en las matrices.

Podemos eliminar la primera fila (de u_1) porque como no conozco $\sigma(0)$ y sí u_1, $\to$ tengo sólo 4 incógnitas $\to$ 4 ecs

Tengo una matriz que no es cuadrada, así que quito la primera columna y paso $k_{21}^1 \times u_1$ restando:

Quedaría: (*) K

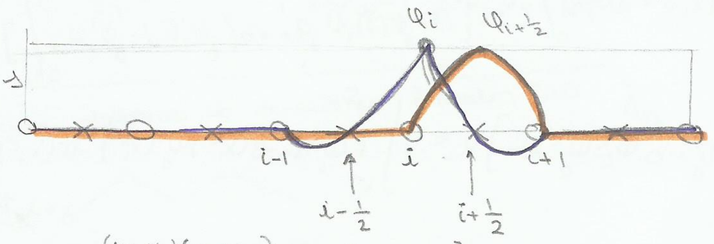
(**) U

(***) $F_j \longrightarrow \begin{pmatrix} f_2^1 + f_1^2 - k_{21}^1 u_1 \\ f_2^2 + f_1^3 + \hat{f} \\ f_2^3 + f_1^4 \\ f_2^4 + P_l u_l \end{pmatrix}$

10/MAR30/2015

4.7- Métodos elementos finitos de grado 2:

10.



$\{\varphi_i\}$

$$\varphi_i = \begin{cases} \dfrac{(x-x_{i-1})(x-x_{i-\frac{1}{2}})}{(x_i-x_{i-1})(x_i-x_{i-\frac{1}{2}})} & \text{en } [x_{i-1}, x_i] \\ \dfrac{(x-x_{i+1})(x-x_{i+\frac{1}{2}})}{(x_i-x_{i+1})(x_i-x_{i+\frac{1}{2}})} & \text{en } [x_i, x_{i+1}] \\ 0 & \text{resto} \end{cases}$$

$$\varphi_{i+\frac{1}{2}} = \begin{cases} \dfrac{(x-x_i)(x-x_{i+1})}{(x_{i+\frac{1}{2}}-x_i)(x_{i+\frac{1}{2}}-x_{i+1})} & \text{en } [x_i, x_{i+1}] \\ 0 & \text{resto} \end{cases}$$

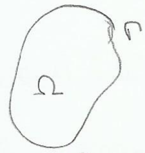
$$u_h(x) = \sum_{i=0}^{n+1} u_i \varphi_i(x) + \sum_{i=0}^{n} u_{i+\frac{1}{2}} \varphi_{i+\frac{1}{2}}(x)$$

* Ejercicios propuestos 8, 9, 10, 11. (TODOS).

TEMA 5. FORMULACIÓN VARIACIONAL DE PROBLEMAS ELÍPTICOS:

5.1. Problema de Dirichlet homogéneo asociado a la ec. Poisson:

$n=2$
$(n=3)$



$$(1)\ \begin{cases} -\Delta u = f & \text{en } \Omega \\ u = 0 & \text{en } \Gamma \end{cases}$$

f test : $v \mid v=0$ en Γ

$$\underbrace{-\int_\Omega \Delta u\, v\, dxdy}_{\text{Green}} = \int_\Omega f v\, dxdy$$

$$\int_\Omega \nabla u \nabla v\, dxdy - \cancel{\int_\Gamma \frac{\partial u}{\partial n} v\, ds} = \int_\Omega f v\, dxdy$$

1D: a ——— b

$$\int_a^b u'' v\, dx = -\int_a^b u'v'\, dx + u'v\Big|_a^b$$

2D:

$$\int_\Omega \Delta u\, v\, dxdy = -\int_\Omega \nabla u \nabla v\, dxdy + \int_\Gamma \frac{\partial u}{\partial n} v\, ds$$

Green: (integral por partes en 2D)

"Hallar $u \in H_0^1(\Omega)$ $\rightarrow$ $(u=0$ en $\Gamma)$

$$\int_\Omega \nabla u \nabla v\, dxdy = \int_\Omega f v\, dxdy \qquad \forall v \in H_0^1(\Omega) \quad [v=0 \text{ en } \Gamma]\text{"}$$

Formulación débil:

17/03/2015

"Teorema 1: La solución del problema débil es única cuando verifique las condiciones de Lax-Milgram"

$$\underbrace{\boxed{\int_\Omega \nabla u \nabla v\, dxdy}}_{a(u,v)} = \underbrace{\boxed{\int_\Omega f v\, dxdy}}_{f(v)}$$

$\exists!\, u \in H_0^1(\Omega) : a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega)$ Lax-Milgram

Apuntes tema 0.

a.) BILINEAL:

$$a(u+w,v) = \int_\Omega \nabla(u+w)\nabla v\, dxdy = \int_\Omega [\nabla u + \nabla w]\nabla v\, dxdy = \int_\Omega \nabla u \nabla v\, dxdy + \int_\Omega \nabla w \nabla v\, dxdy = a(u,v) + a(v,w)$$

$$a(u,v+w) = \int_\Omega \nabla u \nabla(v+w)\, dxdy = a(u,v) + a(u,w)$$

$$a(\lambda u, v) = \int_\Omega \nabla(\lambda u)\nabla v\, dxdy = \lambda \int_\Omega \nabla u \nabla v\, dxdy = \lambda a(u,v)$$ (Como es simétrica, con eso vale, sino tbn $a(u,\lambda v)$)

b.) CONTINUA:

cuando sean $\mathbb{R}$ valor absoluto

$$|a(u,v)| \leq M \|u\|_{1,\Omega} \|v\|_{1,\Omega}$$

$$\left|\int_\Omega \nabla u \nabla v\, dxdy\right| \underset{\text{Cauchy-Schwarz}}{\leq} \left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} \left(\int_\Omega |\nabla v|^2 dxdy\right)^{1/2} = |u|_{1,\Omega} |v|_{1,\Omega} \leq \|u\|_{1,\Omega} \|v\|_{1,\Omega}$$

Siempre $|u|_{1,\Omega} \leq \|u\|_{1,\Omega}$

C.) ELÍPTICA:

$$a(v,v) \geq \alpha \cdot \|v\|^2_{1,\Omega}$$

$$a(v,v) = \int |\nabla v|^2 dxdy = |v|^2_{1,\Omega} \geq \left(\frac{1}{1+C^2}\right) \cdot \|v\|^2_{1,\Omega}$$

$$\|v\|^2_{1,\Omega} = \|v\|^2_{0,\Omega} + |v|^2_{1,\Omega} = (1+C^2)|v|^2_{1,\Omega}$$

Poincaré. $\longrightarrow \|v\|_{0,\Omega} \leq C(\Omega) \cdot |v|_{1,\Omega}$

$f(v)$ a.) lineal. (obvio)

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$$

b.) CONTINUA

$$|f(v)| = \left|\iint_\Omega f v dxdy\right| \leq \left(\int_\Omega |f|^2 dxdy\right)^{\frac{1}{2}} \left(\int |v|^2 dxdy\right)^{\frac{1}{2}} = M \cdot \|v\|_{0,\Omega} \leq M \|v\|_{1,\Omega}$$

$$\|\cdot\|^2_{1,\Omega} = \|\cdot\|^2_{0,\Omega} + |\cdot|^2_{1,\Omega}$$

Se cumplen las 5 hipótesis.

⊛ u y v en el mismo espacio

5.4 PROBLEMA APROXIMADO. MÉTODO DE GALERKIN

$\dim(H^n) = n$ $\quad \{\varphi_i\}_{i=1}^n$ $\quad u_h = \sum_{i=1}^n u_i \varphi_i(x,y)$

$v_h = \varphi_j(x,y)$ $\quad j = 1, \ldots, n$

Hallar $u_h \in H^h$

$$a(u_h, v_h) = f(v_h) \quad \forall v_h \in H^h$$

$$\sum_{i=1}^n a(\varphi_i, \varphi_j) \cdot u_i = f(\varphi_j) \quad j = 1, \ldots, n.$$

mxm (simétrico este caso)

$$\begin{cases} a(\varphi_1,\varphi_1)u_1 + a(\varphi_2,\varphi_1)u_2 + \ldots + a(\varphi_n,\varphi_1)u_n = f(\varphi_1) \\ a(\varphi_1,\varphi_2)u_1 + a(\varphi_2,\varphi_2)u_2 + \ldots + a(\varphi_n,\varphi_2)u_n = f(\varphi_2) \\ - - - - - - - - - - - - \\ a(\varphi_1,\varphi_n)u_n + a(\varphi_2,\varphi_n)u_n + \ldots + a(\varphi_n,\varphi_n)u_n = f(\varphi_n) \end{cases}$$

⊛ Ejerc. prop. 1, 2.

Aplicación Galerkin: Definición $\{\varphi_i\}_{i=1}^n \longrightarrow$ MEF

5.2 PROBLEMA NO HOMOGÉNEO.

Ω Γ

Hallar u

$$(1) \left| \begin{array}{ll} -\Delta u = f & \text{en } \Omega \\ u = g & \text{en } \Gamma \end{array} \right.$$

v: $v = 0$ en Γ

$$-\int_\Omega \Delta u \, v \, dxdy = \int_\Omega f v \, dxdy$$

$v = 0$ en Γ

GREEN:
$$\int_\Omega \nabla u \nabla v \, dxdy - \int_\Gamma \frac{\partial u}{\partial n} v \, dS = \int_\Omega f v \, dxdy$$

5-31

(2) "Hallar $u \in H^1(\Omega)$, $u = g$ en Γ

$$\int_\Omega \nabla u \nabla v \, dxdy = \int_\Omega f v \, dxdy \quad \forall v \in H_0^1(\Omega)"$$

Demostrar:

(*) $\exists! \; u \in H^1(\Omega), u = g$ en Γ $\quad a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega)$ L-M

$u, v \in$ mismo espacio
(pero aquí NO, aunque no podemos aplicarlo)

u_0 $\quad u_0 = g$ en Γ $\quad u \in H^1(\Omega)$ $\quad$ Taller $\Rightarrow u = x+y$ en Γ
$u_0 = x+y$ en Ω y Γ

$w = u - u_0$ $\quad u$ en $\Gamma = 0$ $\quad w \in H_0^1(\Omega)$

$u = w + u_0$

$$\int_\Omega \nabla(w+u_0)\nabla v \, dxdy = \int_\Omega f v \, dxdy \Rightarrow \underbrace{\int_\Omega \nabla \overset{\text{incógnita dw?}}{w} \nabla v \, dxdy}_{a(w,v)} = \underbrace{\int_\Omega f v \, dxdy - \int_\Omega \nabla u_0 \nabla v \, dxdy}_{f(v)}$$

Reformulado (2)

"Hallar $w \in H_0^1(\Omega)$, $a(w,v) = f(v) \quad \forall v \in H_0^1(\Omega)$" $\quad$ v y w mismo espacio $\rightarrow$ aplico L-M

$a(w,v)$ $\quad$ taller $M = \frac{1}{3}$
bilineal (5,1), continua (5,1), H^1-elíptica (5,1)

$f(v)$
lineal: [Obvio]
continua:

$|f(v)| \leq M \|v\|_{1,\Omega}$

$$\Big| \underbrace{\int_\Omega f v \, dxdy}_{(*)} - \int_\Omega \nabla u_0 \nabla v \, dxdy \Big| \leq \Big| \int_\Omega f v \, dxdy \Big| + \Big| \underbrace{\int_\Omega \nabla u_0 \nabla v \, dxdy}_{(**)} \Big| \leq (M+M') \|v\|_{1,\Omega}$$

$$(*) \; \Big| \int_\Omega f v \, dxdy \Big| \overset{C\text{-}S}{\leq} \left(\int_\Omega |f|^2 dxdy \right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dxdy \right)^{\frac{1}{2}} = M \|v\|_{0,\Omega} \leq M \cdot \|v\|_{1,\Omega}$$

$$(**) \; \Big| \int_\Omega \nabla u_0 \nabla v \, dxdy \Big| \overset{C\text{-}S}{\leq} \left(\int_\Omega |\nabla u_0|^2 dxdy \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 dxdy \right)^{\frac{1}{2}} = M' |v|_{1,\Omega} \leq M' \|v\|_{1,\Omega}$$

14/ABRIL/2015

6-2: Aproximación por el mef.

$H^h \subset H^1(\Omega)$ $\{\varphi_i\}_{i=1}^n$ l. indep. $\varphi_i(x_j, y_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\boxed{\varphi_i = a_1 + a_2 x + a_3 y}$ triángulo

$\varphi_i = a_1 + a_2 x + a_3 y + a_4 x \cdot y$ cuadrado.

φ_i pol a trozos.

$$u_h(x,y) = \sum_{i=1}^n \varphi u_i$$

"Halla $u_h \in H^h_1$ $u_h = \hat{u}$ en Γ_{1h}:

$$(3)_h \quad \int_{\Omega_h} \left[k \left(\frac{\partial u_h}{\partial x} \cdot \frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial y} \cdot \frac{\partial v_h}{\partial y} \right) + b u_h v_h \right] dxdy + \int_{\Gamma_{2h}} p u_h v_h ds =$$

$$= \int_{\Omega_h} f v_h dxdy + \int_{\Gamma_{2h}} \gamma v_h ds \quad \forall v_h \in H^h_1 \; v_h = 0 \text{ en } \Gamma_{1h}"$$

$u_h = \sum_{i=1}^n \varphi_i u_i$ $\quad v_h = \varphi_j$ $\quad j = 1, \dots, n$ $\quad$ Comentario: Ω_h Γ_h

cf lados rectos $\to$ curvas?

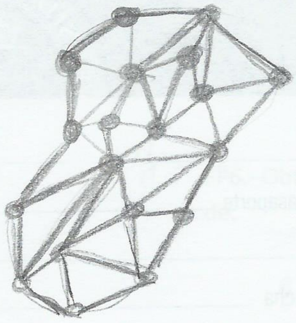
$$\sum_{i=1}^n \left\{ \int \left[k \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y} \cdot \frac{\partial \varphi_j}{\partial y} \right) + b \varphi_i \varphi_j \right] dxdy + \int_{\Gamma_{2h}} p \varphi_i \varphi_j ds \right\} u_i$$

$$F_j = \int_{\Omega_h} f \varphi_j dxdy + \int_{\Gamma_{2h}} \gamma \varphi_j ds \quad j = 1, \dots, n.$$

$$K_{ij} = \int_{\Omega_h} \left[k \left(\frac{\partial \varphi_i}{\partial x} \cdot \frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y} \cdot \frac{\partial \varphi_j}{\partial y} \right) + b \varphi_i \varphi_j \right] dxdy + \int_{\Gamma_{2h}} p \varphi_i \varphi_j ds.$$

$[K_{ij}] \Rightarrow$ matriz de rigidez $\|$ $K_{ij} = 0$

En todos los triángulos φ_i es nula excepto en 2.



Matriz en banda
Lo banda estrecha
(elegir nodos)

TRIANGULO DE PASCAL

GRADO.
0 ——— 1
1 ——— x y
2 ——— x^2 $x \cdot y$ y^2

Tipos de elementos finitos:

a) TRIANGULARES:

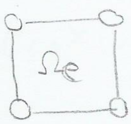
$$u_h^e = a_1 + a_2 x + a_3 y$$

$$u_h^e = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 x \cdot y$$

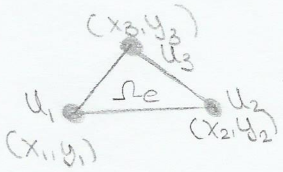
PRODUCTO TENSORIAL

	1	x	x^2
1	1	x	x^2
y	y	xy	yx^2
y^2	y^2	y^2x	y^2x^2

b.) CUADRILÁTEROS:



$$u_h^e = a_1 + a_2 x + a_3 y + a_4 x \cdot y$$



$$u_h^e = a_1 + a_2 x + a_3 y.$$

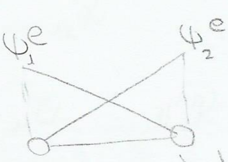
A_c = área triángulo.

$$u_h^e(x_1, y_1) = u_1 = a_1 + a_2 x_1 + a_3 y_1$$
$$u_h^e(x_2, y_2) = u_2 = a_1 + a_2 x_2 + a_3 y_2$$
$$u_h(x_3, y_3) = u_3 = a_1 + a_2 x_3 + a_3 y_3$$

$$a_1 = \frac{1}{2AC}\left[u_1(x_2 y_3 - x_3 y_2) + u_2(x_3 y_1 - x_1 y_3) + u_3(x_1 y_2 - x_2 y_1)\right]$$

$$a_2 = \frac{1}{2A_c}\left[u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2)\right]$$

$$a_3 = \frac{1}{2A_c}\left[u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)\right]$$



Ejercicio propuesto a ESTUDIAR funciones de forma.

$$\{\varphi_1^e, \varphi_2^e, \varphi_3^e\} \longleftarrow \left(\begin{array}{l} \varphi_1^e(x_1, y_1) = \begin{cases} 1 & \text{en } (x_1, y_1) \\ 0 & \text{resto} \end{cases} \\ \varphi_2^e(x_2, y_2) = \begin{cases} 1 & \text{en } (x_2, y_2) \\ 0 & \text{resto.} \end{cases} \end{array} \right.$$

$$u_h^e(x, y) = u_1 \varphi_1^e + u_1 \varphi_2^e + u_3 \varphi_3^e$$

Aproximación local en funciones de forma.

$$\varphi_1^e = \frac{(a_1 + x a_2 + y a_3)}{u_1} = \frac{1}{2A_c}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$\varphi_2^e = \frac{1}{2A_c}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$\varphi_3^e = \frac{1}{2A_c}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

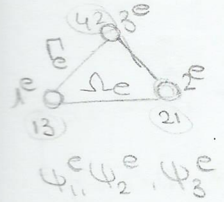
6.3. APROXIMACIÓN LOCAL EN Ω_e

técnica: ① Aproximación local $Ne(3 \text{ ó } 4)$ $Ne \times Ne$

② Ensamblaremos.

③ Condiciones de contorno (D o N)

$\sigma_n = \vec{\sigma}\cdot\vec{n}$ → $k\nabla u$.



u_h^e v_h^e

$\varphi_1^e, \varphi_2^e, \varphi_3^e$

① Localmente, el problema aproximado:

$$\int_{\Omega^e}(k\nabla u_h^e\nabla v_h^e + b u_h^e v_h^e)\,dx\,dy = \int_{\Omega_h} f v_h\,dx\,dy - \int_{\Gamma_e}\sigma_n\cdot v_h^e\,dS$$

En Ω_e $\{\varphi_1^e,\varphi_2^e,\varphi_3^e\}$ $u_h^e=\sum_{i=1}^{Ne}\varphi_i^e u_i^e$,, $v_h^e=\varphi_j^e$ $j=1,\dots,Ne$

$$\sum_{i=1}^{Ne}\boxed{\int_{\Omega e}\left[k\left(\frac{\partial\varphi_i^e}{\partial x}\frac{\partial\varphi_j^e}{\partial x}+\frac{\partial\varphi_i^e}{\partial y}\cdot\frac{\partial\varphi_j^e}{\partial y}\right)+b\varphi_i^e\varphi_j^e\right]dx\,dy}\,u_i^e =$$

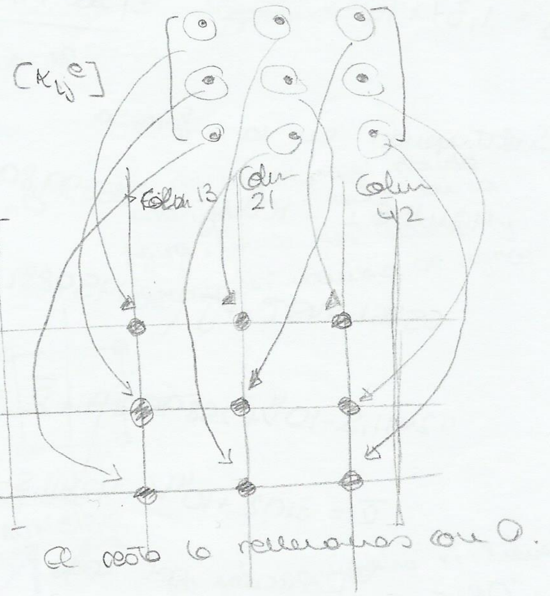
$$=\underbrace{\boxed{\int_{\Omega_h} f\,\varphi_j^e\,dx\,dy}}_{f_j^e}-\underbrace{\boxed{\int_{\Gamma_e}\sigma_n\varphi_j^e\,dS}}_{\sigma_j^e}\qquad j=1,\dots,Ne$$

↓ k_{ij}^e → matriz de rigidez.

$$\sum_{i=1}^{Ne}k_{ij}^e u_i^e = f_j^e-\sigma_j^e \qquad j=1,\dots,Ne$$

$[k_{ij}^e]$ $Ne\times Ne$ $\{f_j^e\}$ $Ne\times 1$ $\{\sigma_j^e\}$ $Ne\times 1$

$[k_{ij}^e]$



② ENSAMBLAJE: "47 sistemas 3×3"

$\mathcal{E}$ sistemas $Ne\times Ne$

LOCAL → GLOBAL

$[k_{ij}^e]$ $Ne\times Ne$ → $[K_{ij}^e]$ $n\times n$

$\{f_j^e\}$ $Ne\times 1$ → $\{F_j^e\}$ $n\times 1$

$\{\sigma_j^e\}$ $Ne\times 1$ → $\{\Sigma_j^e\}$ $n\times 1$

n° nodos

$$\sum_{e=1}^{\mathcal{E}}\left(\sum_{i=1}^{n}K_{ij}^e u_i\right)=\sum_{e=1}^{\mathcal{E}}\left(F_j^e-\Sigma_j^e\right)\qquad j=1,\dots,n$$

ENSAMBLAJE

$$\sum_{e=1}^{\mathcal{E}}K_{ij}^e=\sum_{e=1}^{\mathcal{E}}\int_{\Omega e}\left[k\left(\frac{\partial\varphi_i}{\partial x}\cdot\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\right)+b\varphi_i\varphi_j\right]dx\,dy$$

NODOS → INTERNOS; → EXTERNOS C.-D; → EXTERNOS C-N.

$$\sum_{e=1}^{\mathcal{E}}F_j^e=\sum_{e=1}^{\mathcal{E}}\int_{\Omega e}f\,\varphi_j\,dx\,dy \quad ; \quad \sum_{e=1}^{\mathcal{E}}\Sigma_j^e=\sum_{e=1}^{\mathcal{E}}\int_{\Gamma e}\sigma_n\varphi_j\,dS$$

CONDICIONES DE CONTORNO:

$$\sum_{e=1}^{\mathcal{E}}\Sigma_j^e=\sum_{e=1}^{\mathcal{E}}\int_{\Gamma e}\sigma_n\varphi_j\,dS = S_j^{(0)}+S_j^{(1)}+S_j^{(2)}$$

$$\sum\int_{\Gamma e}\sigma_n\varphi_j\,dS=\sum_{e=1}^{\mathcal{E}}\int_{\Gamma e-\Gamma_h}\sigma_n\varphi_j\,dS+\int_{\Gamma_{1h}}\sigma_n\varphi_j\,dS+\int_{\Gamma_{2h}}\sigma_n\varphi_j\,dS$$

a) $S_j^{(0)}=\int_{\Gamma_1}[\![\sigma_n]\!]\varphi_j\,dS+\int_{\Gamma_2}[\![\sigma_n]\!]\varphi_j\,dS+\int_{\Gamma_3}[\![\sigma_n]\!]\varphi_j\,dS+\int_{\Gamma_4}[\![\sigma_n]\!]\varphi_j\,dS$

a) No hay una fuente puntual en "j" $S_j^{(0)}=0$.

b) hay una fuente $\hat{f}$ $S_j^{(0)}=\hat{f}$

$S_j^{(1)}$ DIRICHLET

$\overline{F_1} = P.S = 1{,}5 \times 10^5 \frac{N}{m^2} \cdot \frac{0{,}6^2 \pi}{4} m^2 = \boxed{42411{,}5 N}$

EXAMEN 1 - FLUIDOS

$\overline{P} = m.g = \rho.V.g = \gamma.V = \boxed{10^4 . V}$

$\rho = \frac{m}{V} \quad m = \rho V$

Sacamos P_2 con Bernoulli

$$\cancel{z_1} + \frac{P_1}{\gamma} + \frac{v_1^2}{2g} = \cancel{z_2} + \frac{P_2}{\gamma} + \frac{v_2^2}{2g}$$ Como están a la misma altura $z_1 = z_2$

$$P_2 = \left(\frac{P_1}{\gamma} + \frac{v_1^2}{2g} - \frac{v_2^2}{2g}\right)\gamma = \left(\frac{1{,}5 \times 10^5}{10^4} + \frac{5^2}{20} - \frac{7{,}2^2}{20}\right) 10^4 = 1{,}5 \times 10^5 + \left(\frac{25 - 51{,}84}{20}\right) \times 10^4 =$$

$$= 136580 \text{ N} = 1{,}37 \times 10^5 \text{N}$$

$\overline{F_2} = 1{,}37 \times 10^5 \times \frac{0{,}5^2 \pi}{4} = \boxed{26899{,}89 \text{ N}}$

Sustituyendo en la fórmula

$$42411{,}5 \vec{i} - 10^4 V \vec{j} + 26899{,}89 \left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right) + \overline{R} = 3102$$

$$55861{,}445 \vec{i} + \vec{j}\left(23295{,}9881 - 10^4 V\right) + \overline{R} = 3102$$

$$42411{,}5 - 10^4 V + 26899{,}89 + \overline{R} = 3102$$

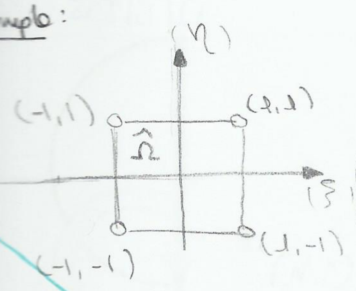
$$\overline{R} = 3102 + 10^4 V - 42411{,}5 - 26899{,}89 = 10^4 V - 66209{,}39$$

Reacción del codo sobre el fluido.

Reacción del fluido sobre el codo $\overline{R}' = -\overline{R} = 66209{,}39 - 10^4 V$

En el dibujo dibujado en naranja

Ejemplo: 8/5/2015



a.) $\hat{\Omega} \to \overbrace{(0,0)\ (1,0)\ (2,2)\ (0,1)}^{\Omega_e}$

$$\hat{\varphi}\begin{cases} x = \sum_{i=1}^{4} x_i \hat{\varphi}_i(\xi,\eta) \\ y = \sum_{i=1}^{4} y_i \hat{\varphi}_i(\xi,\eta) \end{cases}$$

$$\begin{cases} x = 0\cdot\hat{\varphi}_1 + 1\hat{\varphi}_2 + 2\hat{\varphi}_3 + 0\hat{\varphi}_4 = \hat{\varphi}_2 + 2\hat{\varphi}_3 = \frac{1}{4}(1+\xi)(1-\eta) + 2\cdot\frac{1}{4}(1+\xi)(1+\eta) \\ y = 0\cdot\hat{\varphi}_1 + 0\hat{\varphi}_2 + 2\hat{\varphi}_3 + 1\hat{\varphi}_4 = 2\hat{\varphi}_3 + \hat{\varphi}_4 = 2\cdot\frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta) \end{cases}$$

TRANSFORMACIÓN:

$$\begin{cases} x = \frac{1}{4}(3+3\xi+\eta+\xi\eta) \\ y = \frac{1}{4}(3+3\eta+\xi+\xi\eta) \end{cases}$$

$$|J| = \frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi} = \frac{1}{4}\left[(3+\eta)(3+\xi)-(1+\xi)(1+\eta)\right] =$$

$$= \boxed{\frac{1}{2}+\frac{1}{8}(\xi+\eta)=0} \Longrightarrow \begin{cases} 4+\xi+\eta=0 \\ \boxed{\eta = -\xi-4} \end{cases}$$

NO corta a la figura

Es admisible

b.) $\hat{\Omega} \to \overbrace{(1,1)(2,2)(3,0)(3,5)}^{\Omega_e}$

$$\begin{cases} x = 1\hat{\varphi}_1 + 2\hat{\varphi}_2 + 3\hat{\varphi}_3 + 3\hat{\varphi}_4 \\ y = 1\hat{\varphi}_1 + 2\hat{\varphi}_2 + 0\hat{\varphi}_3 + 5\hat{\varphi}_4 \end{cases}$$

---→ transformación como el anterior

$$|J| = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi} = \frac{1}{8}(7-10\xi+10\eta+2\xi\eta)=0.$$

No nos importa por donde pase, sólo si corta con la figura. Damos los valores y miramos si cambia de signos

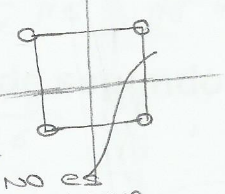
Para (1,1) → $\frac{9}{8}$ (+)

Para (1,−1) → −15 (−)

]→ No haría falta seguir, ya sabemos que cambia de signo (BOLZANO)

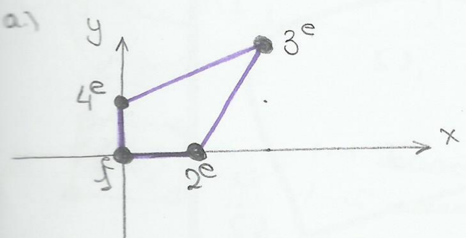
Para (−1,−1) → (+)

Para (−1,1) → (+).

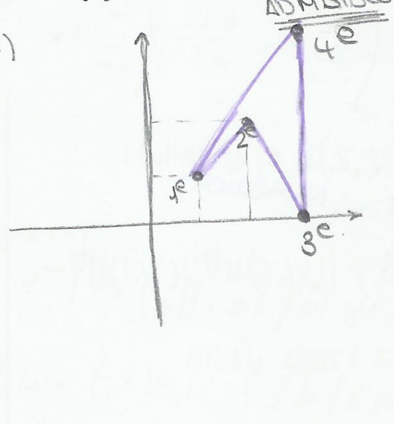


Lo corta a la figura → NO ES ADMISIBLE

a.)



b.)



No son admisibles si tienen ángulos interiores >180°

$$\varepsilon = \frac{Ch^p}{\operatorname{sen}\theta}$$

COMENTARIO. → ángulo interior (+) pequeño

12/5/2015

a) $\varphi_j^e(x,y)$ $\frac{\partial \varphi^e}{\partial x}, \frac{\partial \varphi^e}{\partial y}$

a.) $\to \iint \quad \int_c$ calcularlos!

$T_e \to \{\hat{\Omega}(\xi,\eta) \xrightarrow{T_e} \Omega_e(x,y)\}$

$\hat{\Omega}(\xi(x,y),\eta(x,y)) = \Omega_e$

$\hat{\varphi}_j(\xi,\eta) = \hat{\varphi}_j(\xi(x,y),\eta(x,y)) = \varphi_j^e(x,y)$

$\to T_e; \begin{cases} x = \sum_{k=1}^{N_e} x_k \hat{\varphi}_k(\xi,\eta) \\ y = \sum_{k=1}^{N_e} y_k \hat{\varphi}_k(\xi,\eta) \end{cases}$

$T_e^{-1} \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases}$ $\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$

$\exists |J| \neq 0$.

$T_e \to \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \frac{1}{|J|}\begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$

$T_e^{-1} \to \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$

→ Comparando e igualando:

$$\boxed{\begin{aligned} \frac{\partial \xi}{\partial x} &= \frac{1}{|J|}\frac{\partial y}{\partial \eta} \\ \frac{\partial \xi}{\partial y} &= -\frac{1}{|J|}\frac{\partial x}{\partial \eta} \\ \frac{\partial \eta}{\partial x} &= -\frac{1}{|J|}\cdot\frac{\partial y}{\partial \xi} \\ \frac{\partial \eta}{\partial y} &= \frac{1}{|J|}\cdot\frac{\partial x}{\partial \xi} \end{aligned}}$$

$$\frac{\partial \varphi_j^e}{\partial x} = \frac{\partial \hat{\varphi}_j}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{\varphi}_j}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{1}{|J|}\left(\frac{\partial \hat{\varphi}_j}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial \hat{\varphi}_j}{\partial \eta}\frac{\partial y}{\partial \xi}\right)$$

$$\frac{\partial \varphi_j^e}{\partial x} = \frac{1}{|J|}\left(\frac{\partial \hat{\varphi}_j}{\partial \xi}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{\varphi}_k}{\partial \eta} - \frac{\partial \hat{\varphi}_j}{\partial \eta}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{\varphi}_k}{\partial \xi}\right)$$

$$\frac{\partial \varphi_j^e}{\partial \eta} = \frac{1}{|J|}\left(-\frac{\partial \varphi_j}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial \hat{\varphi}_j}{\partial \eta}\cdot\frac{\partial x}{\partial \xi}\right) = \frac{1}{|J|}\left(-\frac{\partial \hat{\varphi}_j}{\partial \xi}\sum_{k=1}^{N_e} x_k \frac{\partial \hat{\varphi}_k}{\partial \eta} + \frac{\partial \hat{\varphi}_j}{\partial \eta}\sum_{k=1}^{N_e} x_k \frac{\partial \hat{\varphi}_k}{\partial \xi}\right)$$

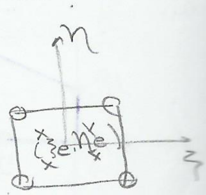
$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi} = \left(\sum_{k=1}^{N_e} x_k \frac{\partial \hat{\varphi}_k}{\partial \xi}\right)\left(\sum_{k=1}^{N_e} y_k \frac{\partial \hat{\varphi}_k}{\partial \eta}\right) - \left(\sum_{k=1}^{N_e} x_k \frac{\partial \hat{\varphi}_k}{\partial \eta}\right)\left(\sum_{k=1}^{N_e} y_k \frac{\partial \hat{\varphi}_k}{\partial \xi}\right)$$

6.7.- Técnicas de integración numérica:

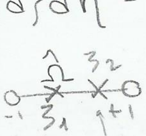
$$\iint_{\hat{\Omega}} G(\xi,\eta)\,d\xi\,d\eta = \sum_{l=1}^{N_3} w_l\, G(\xi_e,\eta_e)$$

n puntos $\begin{cases} N-C \longrightarrow \text{exactos } n-1 \\ G \longrightarrow \text{exactos } 2n-1 \end{cases}$

2×2
3×3
Ω



- 1D - Gauss, nº puntos = 2.



$\int_{-1}^{1} G(\xi)\,d\xi = w_1 G(\xi_1) + w_2 G(\xi_2)$

$\begin{cases} w_1 + w_2 = 2 \\ \xi_1 w_1 + \xi_2 w_2 = 0 \\ \xi_1^2 w_1 + \xi_2^2 w_2 = \frac{2}{3} \\ \xi_1^3 w_1 + \xi_2^3 w_2 = 0 \end{cases}$ $w_1 = w_2 = 1$, $\xi_1 = -\xi_2 = -\frac{1}{\sqrt{3}}$

2D. $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ $w_i = 1$

- n=3 $\int_{-1}^{1} G(\xi)\,d\xi = w_1 G(\xi_1) + w_2 G(\xi_2) + w_3 G(\xi_3)$

$\begin{cases} w_1 + w_2 + w_3 = 2 \\ \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 = 0 \\ \xi_1^2 w_1 + \xi_2^2 w_2 + \xi_3^2 w_3 = \frac{2}{3} \\ \vdots \end{cases}$

$w_2 = \frac{8}{9}$ $w_1 = w_3 = \frac{5}{9}$

$\xi_2 = 0$ $\xi_1 = -\xi_3 = -\sqrt{\frac{3}{5}}$

Tema 7. Problemas parabólicos:

7.1. Aproximación de la ecuación del calor en 1-D mediante diferencias finitas

$u = u(x,t)$

$$\begin{cases} \rho c_p \dfrac{\partial u}{\partial t}(x,t) - k\dfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

EC, CC, CI

$$\Rightarrow \frac{\partial u}{\partial t} - \underbrace{\boxed{\frac{k}{\rho c_p}}}_{\gamma}\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho c_p} f(x,t)$$

$$\frac{\partial x}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = 0.$$

$f(x,t) = 0.$

Comentarios: $\boxed{\text{en 2D}}$ $\dfrac{\partial u}{\partial t} - \gamma\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = 0 \to \dfrac{1}{\rho c_p} f(x,y,t)$

$$\frac{\partial u}{\partial t} - \gamma \Delta u = 0$$

Euler

Crank-Nicolson (o algoritmo)

θ-métodos

$\boxed{\text{C.C. 1D}}$

$$\boxed{\begin{aligned} u(t,0) &= v_1(t) \\ u(t,l) &= v_2(t) \end{aligned}}$$

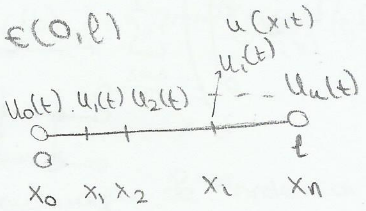
DIRICHLET

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x}(t,0) &= v_1(t) \\ \frac{\partial u}{\partial x}(t,l) &= v_2(t) \end{aligned}}$$

NEUMANN

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0,l)$$

$$\begin{cases} u(0,t) = v_1(t) \\ u(l,t) = v_2(t) \\ u(x,0) = g(x) \end{cases}$$



$h = \dfrac{l}{n}$

Diferencias finitas en x_i

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i,t_j} \simeq \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2}$$

$$\frac{du_i}{dt} = \frac{\gamma}{h^2}\left[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right]$$

$u_0(t) = v_1(t)$; $u_n(t) = v_2(t)$; $u_i(0) = g(x_i)$

$$\{\bar{u}\} = \begin{pmatrix} u_1(t) \\ \vdots \\ u_{n-1}(t) \end{pmatrix} \qquad [E] = \frac{\gamma}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & 1 \cdots & 0 \\ \cdots & & & 1 & -2 \end{bmatrix} \qquad \{v\} = \begin{pmatrix} v_1(t) \\ 0 \\ \vdots \\ v_2(t) \end{pmatrix}$$

$$\begin{cases} \left\{\dfrac{d\bar{u}}{dt}\right\} = [E]\{\bar{u}\} + \{v\} \\ \{\bar{u}_i(0)\} = \{g(x_i)\} \end{cases}$$

C.C. Neumann:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \gamma \dfrac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0,l) \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = v_1(t) \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=l} = v_2(t) \\ u(x,0) = g(x) \end{cases}$$

Euler: $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} \simeq \dfrac{u_1(t) - u_0(t)}{h} = v_1(t)$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=l} \simeq \dfrac{u_{n-1}(t) - u_n(t)}{h} = v_2(t) \Rightarrow$

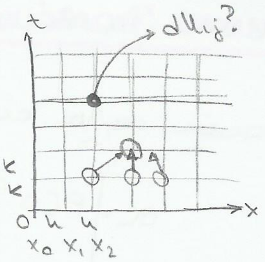
$$\begin{cases} \dfrac{\partial u}{\partial t} - \gamma \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u_0(t) = u_1(t) - h v_1(t) \\ u_n(t) = u_{n-1}(t) - h v_2(t) \\ u(x,0) = g(x) \end{cases}$$

$$\begin{cases} \dfrac{du_i}{dt} = \dfrac{\gamma}{h^2}\left[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right] \\ u_0(t) = u_1(t) - h v_1(t) ;\ u_n(t) = u_{n-1}(t) + h v_2(t) \\ u_i(0) = g(x_i) \end{cases}$$

C.C. DIRICHLET:

$$\left|\begin{array}{l} \dfrac{du_i}{dt} = \dfrac{\gamma}{h^2}\left[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right] \\ u_0(t) = v_1(t) \ ; \ u_n(t) = v_2(t) \\ u_i(0) = g(x) \end{array}\right.$$

$u_{ij} = u(ih, jk)$



$\dfrac{k\gamma}{h^2} = \lambda$

$t = t_j = jk$

Euler $\longrightarrow$ $\left.\dfrac{du_i}{dt}\right|_{t_j} \simeq \dfrac{u_{ij+1} - u_{ij}}{k}$

$$\frac{u_{ij+1} - u_{ij}}{k} = \frac{\gamma}{h^2}\left[u_{i-1j} - 2u_{ij} + u_{i+1j}\right]$$

$$\left|\begin{array}{l} u_{ij+1} = \lambda u_{i-1j} + (1-2\lambda)u_{ij} + \lambda u_{i+1j} \\ u_{0j} = v_1(t_j) \\ u_{nj} = v_2(t_j) \\ u_{i0} = g(x_i) \end{array}\right.$$

MÉTODO EXPLÍCITO

Demostración en el T de que: estabilidad $\lambda \leq \frac{1}{2}$

$\dfrac{k\gamma}{h^2} \leq \dfrac{1}{2}$

$k \leq \dfrac{h^2}{2\gamma}$.

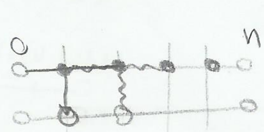
Si ya parto por 4 la (h) separación espacial, me obliga a partir por 16 la temporal (γ)

Para no tener ecuación de estabilidad:

Euler regresiva $\left.\dfrac{du_i}{dt}\right|_{t_j} \simeq \dfrac{u_{ij} - u_{ij-1}}{k} = \lambda u_{i-1j} - 2\lambda u_{ij} + \lambda u_{i+1j}$

$$\left|\begin{array}{l} u_{ij-1} = -\lambda u_{i-1j} + (1-2\lambda)u_{ij} - \lambda u_{i+1j} \\ cc + i \end{array}\right.$$

MÉTODO IMPLÍCITO (+ útil) $\rightarrow$ k no depende de h.



sistema de n-1 ecuaciones

Condición de estabilidad $\lambda \geq 0$

INCISO:

Cuestionario MARTES 26/5. TEMAS 6-7 6 preguntas (2 puntos sobre 3)

EXAMEN (2 partes) $\rightarrow$ Fundamentos teórico-prácticos
$\rightarrow$ Problemas (estilo taller pero con cálculos)

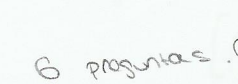
Empezar a preparar ya

19/5/2015

7.2 - APROXIMACIÓN DE LA ECUACIÓN DEL CALOR EN 1D MEDIANTE ELEMENTOS FINITOS:

$h = u(x,t)$

1D



$$(1)\left|\begin{array}{l} \dfrac{\partial u}{\partial t} - \gamma \cdot \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = g(x) \end{array}\right.$$

Dirichlet (espacial)

Sólo nos interesa la x (no la t) en esta primera parte.

$v(x) \mid v(0) = v(l) = 0$ I por partes

$$\int_0^l \frac{\partial u}{\partial t}(x,t)v(x)dx - \int_0^l \gamma \frac{\partial^2 u}{\partial x^2}(x,t)v(x)dx = \int_0^l f(x,t)v(x)dx$$

$$\int_0^l \frac{\partial u}{\partial t}(x,t)v(x)dx + \int_0^l \gamma \frac{\partial u}{\partial x}(x,t)v'(x)dx - \gamma \frac{\partial u}{\partial x}\cancel{(x,t)v(x)}\Big|_0^l = \int_0^l f(x,t)v(x)dx$$

$$\left|\begin{array}{l}\displaystyle\int_0^l \frac{\partial u}{\partial t}(x,t)\,v(x)\,dx + \int_0^l \gamma \frac{\partial u}{\partial x}(x,t)\,v'(x)\,dx = \int_0^l f(x,t)\,v(x)\,dx \\ u(0,t)=u(l,t)=0 \\ u(x,0)=g(x)\end{array}\right.$$

aproximación de u:

Hallar $u_h(x,t)$:

$$\int_0^l \frac{\partial u_h}{\partial t}(x,t)\,v_h(x)\,dt + \int_0^l \gamma \frac{\partial u_h}{\partial x}(x,t)\,v_h'(x)\,dx = \int_0^l f(x,t)\,v_h(x)\,dx.$$

Base (Galerkin) $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ l indep.

$$u_h(x,t) = \sum_{i=1}^n u_i(t)\,\varphi_i(x)$$

$$v_h(x) = \varphi_j(x) \qquad j=1,\dots,n$$

$$\sum_{i=1}^n \left(\int_0^l \varphi_i(x)\varphi_j(x)\,dx\right)\frac{du_i(t)}{dt} + \sum_{i=1}^n \left(\int_0^l \varphi_i'(x)\varphi_j'(x)\,dx\right)u_i(t) = \int_0^l f(x,t)\,\varphi_j(x)\,dx \qquad j=1,\dots,n$$

n ecuaciones diferenciales ordinarias de 1er orden con cond. iniciales

$$M_{ij} = \int_0^l \varphi_i(x)\varphi_j(x)\,dx \;;\quad A_{ij} = \int_0^l \gamma\,\varphi_i'(x)\varphi_j'(x)\,dx \;;\quad \bar{f} = \begin{pmatrix}\int_0^l f(x,t)\varphi_1\,dx \\ \vdots \\ \int_0^l f(x,t)\varphi_n\,dx\end{pmatrix}$$

$$\{\bar{u}\} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t)\end{pmatrix}$$

$$\underbrace{[M_{ij}]}_{\text{matriz de masa}}\cdot\left\{\frac{d\bar{u}}{dt}\right\} + \underbrace{[A_{ij}]}_{[K] \to \text{matriz de rigidez}}\{\bar{u}\} = \{\bar{f}\}$$

NOTA: no hace falta que los puntos sean equidistantes

Mallamos espacio temporal

$$\boxed{\begin{array}{l} u_{ij} = u(ih, jk) \\ = u_i^j \end{array}}$$

$$\left\{\frac{d\bar{u}}{dt}\right\} = \left\{\frac{\bar{u}^{j+1} - \bar{u}^j}{k}\right\}$$ Euler progresivo → Discretizo el instante j con el $j+1$

h: paso respecto a x
k: paso respecto a t

$$(1)\quad [M]\left\{\frac{\bar{u}^{j+1} - \bar{u}^j}{k}\right\} = [A]\{\bar{u}^j\} = \{\bar{f}^j\}$$

error espacial $O(h^2)$
" temporal $O(k)^2$

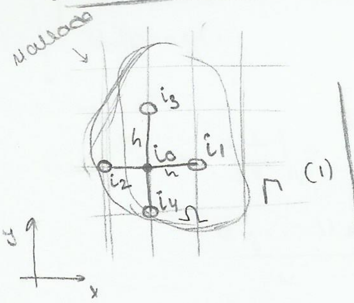
$$\underset{\text{tridiagonal}}{[M]}\{\bar{u}^{j+1}\} = \left(-k[A] + [M]\right)\{\bar{u}^j + k\{\bar{f}^j\}\}$$

Euler regresivo: $(2)\quad [M]\left\{\dfrac{\bar{u}^{j+1} - u^j}{k}\right\} + [A]\{u^{j+1}\} = \{f^{j+1}\}$ Discretizo el instante $j+1$ con el j

$$\frac{(1)+(2)}{2} = [M]\left\{\frac{\bar{u}^{j+1} - \bar{u}^j}{k}\right\} + [A]\left\{\frac{\bar{u}^{j+1} + \bar{u}^j}{2}\right\} = \left\{\frac{\bar{f}^j + \bar{f}^{j+1}}{2}\right\}$$

$$[M] + \frac{k}{2}[A] \{\bar{u}^{j+1}\} = \left([M] - \frac{k}{2}[A]\right)\{u^j\} - \frac{k}{2}\{f + f\}$$

7.3 - APROXIMACIÓN DE PROBLEMAS BIDIMENSIONALES



$u(x,y,t)$

$$(1)\ \Gamma \quad \left| \begin{array}{l} \frac{\partial u}{\partial t} - \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \ \rightarrow \ \frac{\partial u}{\partial t} - \gamma \Delta u = 0 \\ u(x,y,t) = g(t) \quad (x,y) \in \Gamma \\ u(x,y,0) = g(x,y) \end{array} \right.$$

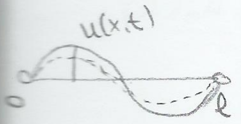
$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{i_0} \rightarrow \begin{cases} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i_0} \simeq \frac{1}{h^2}\left[u_{i_1}(t) - 2u_{i_0}(t) + u_{i_2}(t)\right] \\ \left.\frac{\partial^2 u}{\partial y^2}\right|_{i_0} \simeq \frac{1}{h^2}\left[u_{i_3}(t) - 2u_{i_0}(t) + u_{i_4}(t)\right] \end{cases}$$

$$\simeq \frac{1}{h^2}\left[u_{i_1}(t) + u_{i_2}(t) + u_{i_3}(t) + u_{i_4}(t) - 4u_{i_0}(t)\right]$$

$$\left| \begin{array}{l} \frac{du_i(t)}{dt} = \frac{\gamma}{h^2}\left[u_{i_1}(t) + u_{i_2}(t) + u_{i_3}(t) + u_{i_4}(t) - 4u_{i_0}(t)\right] \\ i = 1, \dots, n \\ u(x_i, y_i, 0) = g(x_i, y_i) \end{array} \right.$$

Tema 8. Problemas hiperbólicos

1. Ecuación de onda. El problema de la cuerda vibrante



$u(x,t) \rightarrow$ posición del punto x de la cuerda en el instante t

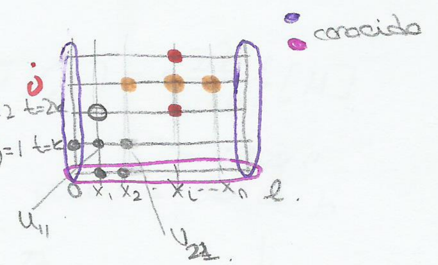
$$(1)\ \left| \ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad x \in (0,l) \quad t>0 \right.$$

$$\text{CC.} \left| \begin{array}{l} u(0,t) = p(t) \\ u(l,t) = q(t) \end{array} \right. \quad \text{Dirichlet no homogénea}$$

$$\text{C.i} \left| \begin{array}{l} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{array} \right.$$

- Discretizar (1) + CC + CI

$$\left| \begin{array}{ll} x_i = ih & i=0,\dots,n \quad h = \frac{l}{n} \\ t_j = jk & \end{array} \right.$$



en (x_i, t_j) $u_{ij} = u(x_i, t_j)$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{i-1j} - 2u_{ij} + u_{i+1j}}{h^2}$$

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{x_i t_j} \simeq \frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{k^2} \ ; \quad \left.\frac{\partial u}{\partial t}\right|_{x_i t_j} \simeq \frac{u_{ij+1} - u_{ij-1}}{2k}$$

$$\frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{k^2} = \frac{u_{i-1j} - 2u_{ij} + u_{i+1j}}{h^2} \qquad \lambda = \frac{k}{h}$$

$$\boxed{u_{ij+1} = \lambda^2 u_{i-1j} + (2-2\lambda^2) u_{ij} + \lambda^2 u_{i+1j} - u_{ij-1}}$$

$$\text{C}_i \left| \begin{array}{l} u_{i0} = f(x) \\ u_{i,1} = u_{i-1} + 2k\,g(x_i) \end{array} \right.$$

$$\text{CC} \left| \begin{array}{l} u_{0j} = p(t_j) \\ u_{nj} = q(t_j) \end{array} \right.$$

$$u_{i,1} = \lambda^2 u_{i-1,0} + (2-2\lambda^2) u_{i,0} + \lambda^2 u_{i+1,0} - u_{ij-1}$$

$$u_{i,1} = u_{ij-1} + g(x_i) \ ; \quad u_{ij-1} = u_{i,1} - 2k\,g(x_i)$$

$$u_{i,1} = \lambda^2 u_{i-1,0} + (2-2\lambda^2) u_{i,0} + \lambda^2 u_{i+1,0} - u_{i,1} + 2k g(x_i)$$

$$u_{i,1} = \frac{\lambda^2}{2} u_{i-1,0} + (1-\lambda^2) u_{i,0} + \frac{\lambda^2}{2} u_{i+1,0} + k g(x_i)$$

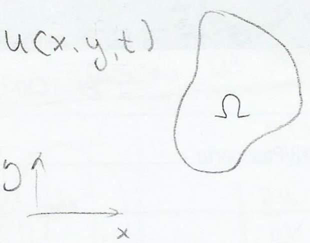
$$\boxed{u_{i,1} = \frac{\lambda^2}{2} f(x_{i-1}) + (1-\lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + k g(x_i)}$$

$$u_{11} = \frac{\lambda^2}{2} f(0) + (1-\lambda^2) f(h) + \frac{\lambda^2}{2} f(2h) + k g(h)$$

$$u_{21} = \frac{\lambda^2}{2} f(h) + (1-\lambda^2) f(2h) + \frac{\lambda^2}{2} f(3h) + k g(2h)$$

8.2. Aproximación de la ecuación de ondas en 2-D por m.e.f

$u(x,y,t)$



$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f(x,y,t) \quad \text{en } \Omega$$

cc $\left[u(x,y,t) = 0 \right.$ en Γ

ci $\left[\begin{array}{l} u(x,y,0) = g(x,y) \\ \dfrac{\partial u}{\partial t}(x,y,0) = h(x,y) \end{array} \right.$

$v(x,y)$ test $\quad v = 0$ en Γ

$$\left.\int_\Omega \frac{\partial^2 u}{\partial t^2} v(x,y)\,dxdy - c^2 \int_\Omega \Delta u\, v(x,y)\,dxdy = \int_\Omega f(x,y,t)\, v(x,y)\,dxdy. \right] \to TG$$

"Hallar u_h: $\displaystyle\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h(x,y)\,dxdy + c^2 \int_\Omega \nabla u_h \nabla v_h\,dxdy = \int_\Omega f(xyt)\, v_h(xy)\,dxdy. \quad \forall v_h \in H_0^1(\Omega)$"

F. base $\quad \varphi_1(xy)\varphi_2(x,y) \text{ --- } \varphi_n(x,y) \qquad u_h(xyt) = \sum_{i=1}^n u_i(t)\varphi_i(xy)$

$$\sum_{i=1}^n \left(\int_\Omega \varphi_i \varphi_j\,dxdy \right) \frac{d^2 u_i}{dt^2} + \sum_{i=1}^n \left(\int_\Omega \left(\frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial y} \right) dxdy \right) \cdot u_i(t) = \int_\Omega f(xyt)\varphi_j\,dxdy \quad (*)$$

$j = 1 \dots n$

$$[M]\left\{\frac{d^2 \bar{u}}{dt}\right\} = [A]\{\bar{u}\} = \{\bar{f}(t)\}$$

n ecs

d. o de 2° orden. $\quad \{\bar{f}(t)\} = \begin{pmatrix} \int_\Omega f\varphi_1\,dxdy \\ \vdots \\ \int_\Omega f\varphi_n\,dxdy \end{pmatrix} \qquad \{\bar{u}\} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$

$j = 1 \dots n$

↓

2n edo de 1er orden.

$u(xy0) = g(x,y) \qquad (*) + $ C. iniciales.

$$\{\bar{u}(0)\} = \begin{pmatrix} g(x_1,y_1) \\ g(x_2 y_2) \\ \vdots \\ g(x_n y_n) \end{pmatrix} \qquad \left\{\frac{d\bar{u}}{dt}(0)\right\} = \begin{pmatrix} h(x_1,y_1) \\ h(x_2,y_2) \\ \vdots \\ h(x_n,y_n) \end{pmatrix}$$